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RELATIVE (NON-)FORMALITY OF THE LITTLE CUBES OPERADS AND THE ALGEBRAIC CERF LEMMA

By VICTOR TURCHIN and THOMAS WILLWACHER

Abstract. It is shown that the operad maps $E_n \to E_{n+k}$ are formal over the reals for $k \geq 2$ and non-formal for $k = 1$. Furthermore we compute the homology of the deformation complex of the operad maps $E_n \to E_{n+1}$, proving an algebraic version of the Cerf lemma.

1. Introduction. We consider the operads of chains E_n of the little n-cubes operads $\mathcal{C}_n$. There are natural embeddings $\mathcal{C}_n \to \mathcal{C}_{n+k}$ for $k \geq 1$, and hence operad maps $E_n \to E_{n+k}$. They induce maps in homology

$$\mathsf{e}_n := H(E_n) \longrightarrow H(E_{n+k}) =: \mathsf{e}_{n+k}.$$

The operad e_n is generated by the two generators of $H(E_n(2)) \cong H(S^{n-1})$. We denote the degree zero generator by $\wedge$ and the degree $n-1$ generator by $[,]$. The map in homology $\mathsf{e}_n \to \mathsf{e}_{n+k}$ above is obtained by sending the product generator to the product and the bracket generator to zero.

A quasi-isomorphism of operad maps $f : \mathcal{P} \to \mathcal{Q}$, $f' : \mathcal{P}' \to \mathcal{Q}'$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\ f\ } & \mathcal{Q} \\ \simeq \downarrow & & \downarrow \simeq \\ \mathcal{P}' & \xrightarrow{\ f'\ } & \mathcal{Q}' \end{array}$$

in which the vertical maps are quasi-isomorphisms. Two maps f and f' are called quasi-isomorphic if they can be related to each other by a zigzag of quasi-isomorphisms. The operad map f is called formal if it is quasi-isomorphic to the induced map $H(f)$ on homology.

We show the following result.

THEOREM 1. *The map $E_n \to E_{n+k}$ is formal over $\mathbb{R}$ for $k \geq 2$ and non-formal over $\mathbb{R}$ for $k = 1$.*

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MICROSOLUTIONS OF DIFFERENTIAL OPERATORS AND VALUES OF ARITHMETIC GEVREY SERIES

By S. FISCHLER and T. RIVOAL

Abstract. We continue our investigation of E-operators, in particular their connection with G-operators; these differential operators are fundamental in understanding the diophantine properties of Siegel's E and G-functions. We study in detail microsolutions (in Kashiwara's sense) of Fuchsian differential operators, and apply this to the construction of bases of solutions at 0 and ∞ of any E-operator from microsolutions of a G-operator; this provides a constructive proof of a theorem of André. We also focus on the arithmetic nature of connection constants and Stokes constants between different bases of solutions of E-operators. For this, we introduce and study in details an arithmetic (inverse) Laplace transform that enables one to get rid of transcendental numbers inherent to André's original approach. As an application, we define a set of special values of arithmetic Gevrey series, and discuss its conjectural relation with the ring of exponential periods of Kontsevich-Zagier.

1. Introduction. In this paper, we continue our investigation of the arithmetic properties of certain differential operators related to E and G-functions. Throughout the paper we fix a complex embedding of $\overline{\mathbb{Q}}$ and let $\mathbb{N} = \{0,1,2,\ldots\}$. To begin with, let us recall the following definition, essentially due to Siegel.

Definition 1. A G-function G is a formal power series $G(z) = \sum_{n=0}^{\infty} a_n z^n$ such that the coefficients a_n are algebraic numbers and there exists $C > 0$ such that:

(i) the maximum of the moduli of the conjugates of a_n is $\leq C^{n+1}$ for any n.

(ii) there exists a sequence of rational integers d_n, with $|d_n| \leq C^{n+1}$, such that $d_n a_m$ is an algebraic integer for all $m \leq n$.

(iii) $G(z)$ satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

An E-function is defined similarly, as $E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ with the same assumptions (i), (ii), (iii) (with $E(z)$ instead of $G(z)$ there).

A minimal differential equation satisfied by a given G-function is called a G-operator. Any E-function is solution of an E-operator (not necessarily minimal) obtained as the Fourier-Laplace transform of a G-operator. In [4], André set the foundations of the theory of E-operators; he proved in particular that 0 and ∞ are the only possible singularities, with rational exponents, and that 0 is a regular one. He also constructed two special bases of solutions of any E-operator at 0 and ∞ respectively, and obtained a fundamental duality between these bases. A basic

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A MODEL OF VISCOELASTICITY WITH TIME-DEPENDENT MEMORY KERNELS

By MONICA CONTI, VALERIA DANESE, CLAUDIO GIORGI, and VITTORINO PATA

Abstract. We consider the model equation arising in the theory of viscoelasticity

$$\partial_{tt}u - h_t(0)\Delta u - \int_0^\infty h_t'(s)\Delta u(t-s)\mathrm{d}s + f(u) = g.$$

Here, the main feature is that the memory kernel $h_t(\cdot)$ depends on time, allowing for instance to describe the dynamics of aging materials. From the mathematical viewpoint, this translates into the study of dynamical systems acting on time-dependent spaces, according to the newly established theory of Di Plinio, Duane, and Temam. In this first work, we give a proper notion of solution, and we provide a global well-posedness result. The techniques naturally extend to the analysis of the longterm behavior of the associated process, and can be exported to cover the case of general systems with memory in presence of time-dependent kernels.

1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For any given $\tau \in \mathbb{R}$, we consider for $t > \tau$ the evolution equation arising in the theory of uniaxial deformations in isothermal viscoelasticity (see e.g., [3, 17, 28])

$$\partial_{tt}u - h(0)\Delta u - \int_0^\infty h'(s)\Delta u(t-s)\mathrm{d}s + f(u) = g, \tag{1.1}$$

subject to the homogeneous Dirichlet boundary condition

$$u(t)_{|\partial\Omega} = 0. \tag{1.2}$$

The unknown variable $u = u(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ describes the *axial displacement field* relative to the reference configuration of a viscoelastic body occupying the volume Ω at rest, and is interpreted as an initial datum for $t \leq \tau$, where it need not solve the equation. Here, $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear term, $g = g(x) : \Omega \to \mathbb{R}$ an external force, and the convolution (or memory) kernel h is a function of the form

$$h(s) = k(s) + k_\infty,$$

where k is a (nonnegative) convex summable function. The values $h(0) > k_\infty > 0$ represent the *instantaneous elastic modulus*, and the *relaxation modulus* of the

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OPTIMAL BOUNDS FOR THE VOLUMES OF KÄHLER-EINSTEIN FANO MANIFOLDS

By KENTO FUJITA

Abstract. We show that any n-dimensional Ding semistable Fano manifold X satisfies that the anti-canonical volume is less than or equal to the value $(n+1)^n$. Moreover, the equality holds if and only if X is isomorphic to the n-dimensional projective space. Together with a result of Berman, we get the optimal upper bound for the anti-canonical volumes of n-dimensional Kähler-Einstein Fano manifolds.

Contents.

1. Introduction. An n-dimensional smooth complex projective variety X is said to be a *Fano manifold* if the anti-canonical divisor $-K_X$ is ample. If $n \leq 3$, then the anti-canonical volume $((-K_X)^{\cdot n})$ is less than or equal to $(n+1)^n$, and the equality holds if and only if X is isomorphic to the projective space $\mathbb{P}^n$ by [Isk77, Isk78, MM81]. However, if $n \geq 4$, there exists an n-dimensional Fano manifold X such that $((-K_X)^{\cdot n}) > (n+1)^n$ holds (see [IP99, p. 128] for example). Recently, Berman and Berndtsson [BB11] conjectured that, if X admits *Kähler-Einstein metrics*, then the value $((-K_X)^{\cdot n})$ would be less than or equal to $(n+1)^n$. In fact, if X is toric, then the conjecture is true by [BB11, Theorem 1] and [NP14, Proposition 1.3]. Moreover, Berman and Berndtsson [BB12] proved the above conjecture under the assumption that X admits a $\mathbb{G}_m$-action with finite number of fixed points.

The purpose of this article is to refine the result [BB12] in full generality. The following is the main result in this article.

THEOREM 1.1. (Main Theorem) *Let X be an n-dimensional Fano manifold admitting Kähler-Einstein metrics. If $((-K_X)^{\cdot n}) \geq (n+1)^n$, then $X \simeq \mathbb{P}^n$.*

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GLOBAL REGULARITY FOR THE FREE BOUNDARY IN THE OBSTACLE PROBLEM FOR THE FRACTIONAL LAPLACIAN

By BEGOÑA BARRIOS, ALESSIO FIGALLI, and XAVIER ROS-OTON

Abstract. We study the regularity of the free boundary in the obstacle problem for the fractional Laplacian under the assumption that the obstacle φ satisfies $\Delta\varphi \le 0$ near the contact region. Our main result establishes that the free boundary consists of a set of regular points, which is known to be a $(n-1)$-dimensional $C^{1,\alpha}$ manifold by the results obtained by Caffarelli, Salsa, and Silvestre (*Invent. Math.* 2008), and a set of singular points, which we prove to be contained in a union of k-dimensional C^1-submanifold, $k = 0,\dots,n-1$. Such a complete result on the structure of the free boundary, proved by L. A. Caffarelli (*Acta Math.* 1977 and *J. Fourier Anal. Appl.* 1998), was known only in the case of the classical Laplacian, and it is new even for the Signorini problem (which corresponds to the particular case of the $\frac{1}{2}$-fractional Laplacian). A key ingredient behind our results is the validity of a new non-degeneracy condition $\sup_{B_r(x_0)}(u-\varphi) \ge cr^2$, valid at all free boundary points x_0.

1. Introduction and main results.

1.1. The obstacle problem for the fractional Laplacian. Given a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the obstacle problem for the fractional Laplacian can be written as

$$\begin{cases} \min\{u-\varphi, (-\Delta)^s u\} = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.1}$$

where

$$(-\Delta)^s u(x) = c_{n,s}\,\mathrm{PV}\int_{\mathbb{R}^n} \big(u(x)-u(x+z)\big)\frac{dz}{|z|^{n+2s}}, \qquad s\in(0,1),$$

is the fractional Laplacian.

These kinds of obstacle problems naturally appear when studying the optimal stopping problem for a stochastic process, and in particular they are used in the pricing of American options. Indeed, the operator $(-\Delta)^s$ corresponds to the case where the underlying stochastic process is a stable radially symmetric Lévy process. We provide in the appendix a brief informal description of the optimal stopping problem and its relation to (1.1).

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ON THE MAGNITUDES OF COMPACT SETS IN EUCLIDEAN SPACES

By JUAN ANTONIO BARCELÓ and ANTHONY CARBERY

Abstract. The notion of the magnitude of a metric space was introduced by Leinster and developed in works by Leinster, Meckes and Willerton, but the magnitudes of familiar sets in Euclidean space are only understood in relatively few cases. In this paper we study the magnitudes of compact sets in Euclidean spaces. We first describe the asymptotics of the magnitude of such sets in both the small- and large-scale regimes. We then consider the magnitudes of compact convex sets with nonempty interior in Euclidean spaces of odd dimension, and relate them to the boundary behaviour of solutions to certain naturally associated higher order elliptic boundary value problems in exterior domains. We carry out calculations leading to an algorithm for explicit evaluation of the magnitudes of balls, and this establishes the convex magnitude conjecture of Leinster and Willerton in the special case of balls in dimension three. In general the magnitude of an odd-dimensional ball is a rational function of its radius, thus disproving the general form of the Leinster-Willerton conjecture. In addition to Fourier-analytic and PDE techniques, the arguments also involve some combinatorial considerations.

1. Introduction. Motivated by considerations of a category-theoretic nature, Leinster [11] has introduced the notion of the *magnitude* of a metric space. Magnitude is an important new numerical invariant of a metric space which shares some of the more abstract properties of the Euler characteristic of a category (or of a topological space), and indeed both can be seen as special cases of the notion of the Euler characteristic or magnitude of an *enriched category*. In particular, the inclusion-exclusion principle enjoyed by the Euler characteristic provides important motivation for the hoped-for properties of magnitude. More generally, magnitude is designed to capture the "essential size" of a metric space in a more subtle way than cruder measures such as cardinality or diameter, and at the same time it will also contain further significant geometric information concerning the space. For a much more detailed discussion of these issues see [11, 12, 13, 20].

Leinster's definition of the magnitude of a finite metric space bears close resemblance to notions of a potential-theoretic nature, and Meckes [16, 17] has developed this perspective to the point where a tractable definition of the magnitude of a positive-definite compact metric space can now be given in terms analogous to those of classical capacity. This provides the starting point for our investigations.

Before describing our results, we give a little more informal background on magnitude in order that our contributions can be placed in context.

1.1. Definitions of magnitude and connection with capacity. Given a finite metric space (X,d), Leinster [11] defined its magnitude as the value

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KATO'S EULER SYSTEM AND THE MAZUR-TATE REFINED CONJECTURE OF BSD TYPE

By KAZUTO OTA

Abstract. Mazur and Tate proposed a conjecture which compares the Mordell-Weil rank of an elliptic curve over $\mathbb{Q}$ with the order of vanishing of Mazur-Tate elements, which are analogues of Stickelberger elements. Under some relatively mild assumptions, we prove this conjecture. Our strategy of the proof is to study divisibility of certain derivatives of Kato's Euler system.

Contents.

1. Introduction.

1.1. The Mazur-Tate refined conjecture of BSD type. Mazur-Tate [20] proposed a *refined conjecture of BSD type*, which predicts mysterious relations between arithmetic invariants of an elliptic curve E over $\mathbb{Q}$ and Mazur-Tate elements constructed from modular symbols. Mazur-Tate elements are analogues of Stickelberger elements and refine the p-adic L-function of E. As the Birch and Swinnerton-Dyer conjecture does, the Mazur-Tate refined conjecture of BSD type consists of two parts. One compares the Mordell-Weil rank with the "order of vanishing" of Mazur-Tate elements (the rank-part). The other describes the "leading coefficients" of the elements. The aim of this paper is to prove the rank-part under some mild assumptions. Now, we explain this part more precisely (see Section 2 for the other part).

We let S be a positive integer and put $G_S = \mathrm{Gal}(\mathbb{Q}(\zeta_S)/\mathbb{Q})$, where ζ_S is a primitive S-th root of unity. The *Mazur-Tate element* θ_S is an element of $\mathbb{Q}[G_S]$ such that for every character χ of G_S, the evaluation $\chi(\theta_S)$ equals the algebraic part

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STABILITY OF THE BRASCAMP-LIEB CONSTANT AND APPLICATIONS

By JONATHAN BENNETT, NEAL BEZ, TARYN C. FLOCK, and SANGHYUK LEE

Abstract. We prove that the best constant in the general Brascamp-Lieb inequality is a locally bounded function of the underlying linear transformations. As applications we deduce certain very general Fourier restriction, Kakeya-type, and nonlinear variants of the Brascamp-Lieb inequality which have arisen recently in harmonic analysis.

1. Introduction. The celebrated Brascamp-Lieb inequality, which simultaneously generalizes many important multilinear inequalities in analysis, including the Hölder, Loomis-Whitney and Young convolution inequalities, takes the form

$$\int_H \prod_{j=1}^m (f_j \circ L_j)^{p_j} \le C \prod_{j=1}^m \left(\int_{H_j} f_j \right)^{p_j}. \tag{1}$$

Here m denotes a positive integer, H and H_j denote euclidean spaces of finite dimensions n and $n_j \le n$ respectively, equipped with Lebesgue measure for each $1 \le j \le m$. The maps $L_j : H \to H_j$ are surjective linear transformations, and the exponents $0 \le p_j \le 1$ are real numbers. This inequality is often referred to as multilinear, since it is equivalent to

$$\int_H \prod_{j=1}^m f_j \circ L_j \le C \prod_{j=1}^m \|f_j\|_{L^{q_j}(H_j)} \tag{2}$$

where $q_j = p_j^{-1}$ for each j.

Following the notation introduced in [10] we denote by $\mathrm{BL}(\mathbf{L},\mathbf{p})$ the smallest constant C for which (1) holds for all nonnegative input functions $f_j \in L^1(\mathbb{R}^{n_j})$, $1 \le j \le m$. Here $\mathbf{L}$ and $\mathbf{p}$ denote the m-tuples $(L_j)_{j=1}^m$ and $(p_j)_{j=1}^m$ respectively. We refer to $(\mathbf{L},\mathbf{p})$ as the *Brascamp-Lieb datum*, and $\mathrm{BL}(\mathbf{L},\mathbf{p})$ as the *Brascamp-Lieb constant*. To avoid completely degenerate cases, where $\mathrm{BL}(\mathbf{L},\mathbf{p})$ is easily seen to be infinite, it is natural to restrict attention to data $(\mathbf{L},\mathbf{p})$ for which

$$\bigcap_{j=1}^m \ker L_j = \{0\}.$$

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